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Quantal Poincaré-Cartan Integral Invariant for Field Theory

Zhang Ying[1,3] and Li Ziping[1,2]

On the basis of the phase-space generating function of Green function for a system with a regular/singular Lagrangian, the quantal Poincaré-Cartan integral invariant (PCII) for field theory is derived. This PCII is equivalent to the quantal canonical equations. For this case in which the Jacobian of the transformation does not equal to unity, the quantal PCII can still be derived. This case is different from the quantal first Noether theorem. The quantal PCII connected with canonical equations and canonical transformation is also discussed.

KEY WORDS: field theory; path integral; quantal Poincaré-Cartan integral invariant.

1. INTRODUCTION

The motion of microcosmic particles can be described by quantum theory. For the transition from the classical theories to the quantum theories, the path-integral quantization can be used as well as the formulation of canonical (operator) quantization. The two formulations are equivalent. In the formulation of path-integral quantization, the main ingredient is the classical action together with the measure in the space of field configurations. Thus path integrals provide a useful tool for studying the quantum symmetries of a system. The phase-space path integrals are more fundamental than the configuration-space path integrals (Mazrabi, 1978).

The PCII plays an important role in mechanics and field theories since, from its invariance it follows that the equations of motion of the dynamical system are canonical equations. It can be treated as a fundamental dynamical principle in classical theories. For a system with a regular/singular Lagrangian, the PCII

[1] Department of Applied Physics, Beijing Polytechnic University, Beijing 100022, Peoples Republic of China.
[2] Chinese Center of Advanced Science and Technology (CCAST) (World Laboratory), Beijing 100080, Peoples Republic of China.
[3] To whom correspondence should be addressed at Department of Applied Physics, Beijing Polytechnic University, Beijing 100022, Peoples Republic of China; e-mail: zhangying792002@yahoo.com.cn.

is equivalent to the classical canonical equations. The PCII has been extended to the nonholonomic system at the classical level (Gantmacher, 1970; Li and Li, 1990; Li and Wu, 1994; Mei *et al.*, 1991). The generalized PCII for a system with a higher order Lagrangian was also studied (Li, 1993; Li and Jiang, 2002). Howerver, these investegation of the PCII for the system are developed at the classical level (Benavent and Gomis, 1979; Dominici and Gomis, 1980). It needs further study whether they hold true or not at the quantum level.

In Li and Li (2001), the quantal PCII for the system with finite degrees of freedom has been developed, but the symbol of ground state still appears in the PCII. In this paper, based on the phase-space generating function of Green function for a system with a regular/singular Lagrangian, the quantal PCII in field theories is derived and the symbol of the ground state is eliminated. It is pointed out that the quantal PCII is equivalent to the quantal canonical equations. The quantal PCII connected with canonical equations and canonical transformation is also discussed. For this case in which the Jacobian of the transformation is not equal to unity, the quantal PCII can be still derived. This case is different from the quantal first Noether theorem. The comparisons of the results at the quantum level and those in classical theories are discussed.

2. THE QUANTAL PC INTEGRAL INVARIANT

Let us first consider a system with a regular Lagrangian described by the field variables $\psi^\alpha(x)(\alpha = 1, 2, \ldots, n)$, $\quad x = (x_0, x_i), (x_0 = t, i = 1, 2, 3)$. The motion of the field is described by a Lagrangian density $L(\psi^\alpha, \psi^\alpha_\mu)$. Introducing the exterior $J_\alpha(x)$ and $K^\alpha(x)$ with respect to the fields $\psi^\alpha(x)$ and their canonical momentum $\pi_\alpha(x)$, respectively. The generating functional of the Green function in phase space for this system can be written as (Li and Jiang, 2002)

$$Z[J, K] = \int \mathcal{D}\psi^\alpha \mathcal{D}0\pi_\alpha \exp\left\{i\left[I^P + \int d^4x(J_\alpha\psi^\alpha + K^\alpha\pi_\alpha)\right]\right\} \tag{1}$$

We consider the space coordinates x_i to be a fixed parameter (Musicki, 1978). A “curve” in the phase space is defined by

$$\psi^\alpha = \psi^\alpha(t, \theta), \quad \pi_\alpha = \pi_\alpha(t, \theta) \tag{2}$$

where θ is a parameter. Let us consider the infinitesimal transformation in extended phase space which arise from the change of the parameter θ (x_i is fixed).

$$\begin{cases} t \to t' = t + \Delta t(\theta) \\ \psi^\alpha(t, x_i) \to \psi'^\alpha(t', x_i) = \psi^\alpha(t, x_i) + \Delta\psi^\alpha(t, x_i, \theta) \\ \pi_\alpha(t, x_i) \to \pi'_\alpha(t', x_i) = \pi_\alpha(t, x_i) + \Delta\pi_\alpha(t, x_i, \theta) \end{cases} \tag{3}$$

where θ satisfy

$$\psi'^{\alpha}(t, x_i, 0) = \psi^{\alpha}(t, x_i), \quad \pi'_{\alpha}(t, x_i, 0) = \pi_{\alpha}(t, x_i) \tag{4}$$

Under the transformation (3), the variation of the canonical action is given by

$$\Delta I^p = \int d^4x \left(\frac{\delta I^P}{\delta \psi^{\alpha}} \delta \psi^{\alpha} + \frac{\delta I^p}{\delta \pi_{\alpha}} \delta \pi^{\alpha} \right) + \int d^4x \left\{ \partial_{\mu}[(\pi_{\alpha}\dot{\psi}^{\alpha} - \mathcal{H}_c)\Delta x^{\mu}] + \frac{d}{dt}(\pi_{\alpha}\delta\psi^{\alpha}) \right\} \tag{5}$$

where

$$\frac{\delta I^p}{\delta \psi^{\alpha}} = -\dot{\pi}_{\alpha} - \frac{\delta \mathcal{H}_c}{\delta \psi^{\alpha}}, \quad \frac{\delta I^p}{\delta \pi_{\alpha}} = \dot{\psi}^{\alpha} - \frac{\delta \mathcal{H}_c}{\delta \pi_{\alpha}} \tag{6a}$$

$$\mathcal{H}_c = \int_V dx^3 \mathcal{H}_c = \int_V dx^3(\pi_{\alpha}\dot{\psi}^{\alpha} - \mathcal{L}^p) \tag{6b}$$

The relations between the substantial variations $\delta\psi^{\alpha}$, $\delta\pi_{\alpha}$ and the total variations $\Delta\psi^{\alpha}$, $\Delta\pi_{\alpha}$ are given by

$$\delta\psi^{\alpha} = \Delta\psi^{\alpha} - \psi^{\alpha}_{\mu}\delta x^{\mu} = \Delta\psi^{\alpha} - \psi^{\alpha}_{,0}\Delta x^0 \tag{7a}$$

$$\delta\pi_{\alpha} = \Delta\pi_{\alpha} - \pi^{\delta}_{\alpha,\mu}x^{\mu} = \Delta\pi_{\alpha} - \pi_{\alpha,0}\Delta x^0 \tag{7b}$$

Let it be supposed that the Jacobian of the transformation (3) of the field variables is $\bar{J}(\theta) = 1 + J_1(\theta)(\bar{J}(0) = 1)$. The smoothed function $J_1(\theta)$ can be expressed by using a total differential function $Q(\theta)$ i.e. $J_1(\theta) = dQ(\theta)/d\theta$. Under the transformation (3), the generating function is invariant which can be written as

$$Z[J, K] = \int \mathcal{D}\psi^{\alpha}\mathcal{D}\pi_{\alpha} \left\{ 1 + dQ/d\theta + i \int dx^4 \left[\left(\frac{\delta I^P}{\delta\psi^{\alpha}} + J_{\alpha} \right) \delta\psi^{\alpha} + \left(\frac{\delta I^p}{\delta\pi_{\alpha}} + K^{\alpha} \right) \delta\pi_{\alpha} \right] + i \int dx^4 \left\{ \partial_{\mu}[(\pi_{\alpha}\psi^{\alpha} - \mathcal{H}_c)\Delta x^{\mu}] + \frac{d}{dt}(\pi_{\alpha}\delta\psi^{\alpha}) \right\} \right\} \cdot \exp\left\{ i \left[I^p + \int dx^4(J_{\alpha}\psi^{\alpha} + K^{\alpha}\pi_{\alpha}) \right] \right\} \tag{8}$$

From the invariance of the generating functional (1) under the transformation (3), one obtains

$$\int \mathcal{D}\psi^{\alpha}\mathcal{D}\pi_{\alpha} \left\{ dQ/d\theta + i \int dx^4 \left[\left(\frac{\delta I^P}{\delta\psi^{\alpha}} + J_{\alpha} \right) \delta\psi^{\alpha} + \left(\frac{\delta I^p}{\delta\pi_{\alpha}} + K^{\alpha} \right) \delta\pi_{\alpha} \right] + i \int dx^4 \left\{ \partial_{\mu}[(\pi_{\alpha}\dot{\psi}^{\alpha} - \mathcal{H}_c)\Delta x^{\mu}] + \frac{d}{dt}(\pi_{\alpha}\delta\psi^{\alpha}) \right\} \right\} \cdot \exp\left\{ i[I^p + \int_{dx^4}(J_{\alpha}\psi^{\alpha} + K^{\alpha}\pi_{\alpha})] \right\} = 0 \tag{9}$$

Functionally differentiating (9) with respect to J_α, one obtains

$$\int \mathcal{D}\psi^\alpha \mathcal{D}\pi_\alpha \left(\left\{idQ/d\theta - \int dx^4 \left[\left(\frac{\delta I^P}{\delta\psi^\alpha} + J_\alpha\right)\delta\psi^\alpha + \left(\frac{\delta I^p}{\delta\pi_\alpha} + K^\alpha\right)\delta\pi_\alpha\right]\right.\right.$$
$$\left.\left. - \int dx^4 \left\{\partial_\mu\left[(\pi_\alpha\dot{\psi}^\alpha - \mathcal{H}_c)\Delta x^\mu\right] + \frac{d}{dt}(\pi_\alpha\delta\psi^\alpha)\right\}\right\}\cdot\psi^\alpha(x_1) + iN^{\alpha\sigma}\right)$$
$$\cdot\exp\left\{i\left[I^p + \int dx^4(J_\alpha\psi^\alpha + K^\alpha\pi_\alpha)\right]\right\} = 0 \tag{10}$$

where

$$N^{\alpha\sigma} = \delta\psi^\alpha = \Delta\psi^\alpha - \psi^\alpha_{,0}\Delta x^0 \tag{11}$$

Then, functionally differentiating (9) with respect to J_α a total of n times, one gets

$$\int \mathcal{D}\psi^\alpha \mathcal{D}\pi_\alpha \left(\left\{idQ/d\theta - \int dx^4 \left[\left(\frac{\delta I^P}{\delta\psi^\alpha} + J_\alpha\right)\delta\psi^\alpha + \left(\frac{\delta I^p}{\delta\pi_\alpha} + K^\alpha\right)\delta\pi_\alpha\right]\right.\right.$$
$$\left. - \int dx^4\{\partial_\mu[(\pi_\alpha\dot{\psi}^\alpha - \mathcal{H}_c)\Delta x^\mu] + \frac{d}{dt}(\pi_\alpha\delta\psi^\alpha)\}\right\}\cdot\psi^\alpha(x_1)\psi^\alpha(x_2)\cdots\psi^\alpha(x_n)$$
$$\left. + i\sum_j \psi^\alpha(x_1)\cdots\psi^\alpha(x_{j-1})\psi^\alpha(x_{j+1})\cdots\psi^\alpha(x_n)\cdot N^{\alpha\sigma} + iN^{\alpha\sigma}\right)$$
$$\cdot\exp\left\{i\left[I^p + \int dx^4(J_\alpha\psi^\alpha + K^\alpha\pi_\alpha)\right]\right\} = 0 \tag{12}$$

Let $J = K = 0$ in (12), one gets (Young, 1987)

$$\langle 0|T^*\left[-idQ/d\theta + \int d^4x\left(\frac{\delta I^p}{\delta\psi^\alpha}\delta\psi^\alpha + \frac{\delta I^p}{\delta\pi_\alpha}\delta\pi_\alpha\right)\right.$$
$$\left. + \int_{t_1}^{t_2} D\int_V d^3x(\pi_\alpha\Delta\psi^\alpha - \mathcal{H}_c\Delta t)\right]\psi^\alpha(x_1)\psi^\alpha(x_2)\cdots\psi^\alpha(x_n)|0\rangle$$
$$- i\langle 0|\sum_j \psi^\alpha(x_1)\cdots\psi^\alpha(x_{j-1})\psi^\alpha(x_{j+1})\cdots\psi^\alpha(x_n)\cdot N^{\alpha\sigma} + N^{\alpha\sigma}|0\rangle = 0 \tag{13}$$

Where the symbol T^* stands for the covariantized T product (Young, 1987), $D = \frac{d}{dt}$. From (11) one can see that the smoothed function of θ can also be expressed as

$$\langle 0|\sum_j \psi^\alpha(x_1)\cdots\psi^\alpha(x_{j-1})\psi^\alpha(x_{j+1})\cdots\psi^\alpha(x_n)\cdot N^{\alpha\sigma}\,|0\rangle = f(\theta) = dF(\theta)/d\theta \tag{14a}$$

$$\langle 0|\,N^{\alpha\sigma}\,|0\rangle = g(\theta) = dG(\theta)/d\theta \tag{14b}$$

Fixing t and letting $t_1, t_2, \ldots, t_m \to +\infty, t_{m+1}, t_{m+2}, \ldots, t_n \to -\infty$, noting that $\psi^\alpha(\vec{x}, -\infty)\,|0\rangle = |1, in\rangle$, $\langle 0|\,\psi^\alpha(\vec{x}, \infty) = \langle out, 1|$ and using the reduction formular (Young, 1987), one can write the expression (13) as

$$\langle out, m|\, T^* \left[\int d^4x \left(\frac{\delta I^p}{\delta\psi^\alpha}\delta\psi^\alpha + \frac{\delta I^p}{\delta\pi_\alpha}\delta\pi_\alpha\right) |out, m\rangle + \langle out, m| \right.$$
$$\left. \times \int_V d^3x(\pi_\alpha \Delta\psi^\alpha - \mathcal{H}_c \Delta t)\right] |n - m, in\rangle|_{t_1} - \langle out, m|$$
$$\times \int_V d^3x(\pi_\alpha \Delta\psi^\alpha - \mathcal{H}_c \Delta t)\, |n - m, in\rangle|_{t_2} = i\{dF/d\theta$$
$$+ dG/d\theta + \langle out, m|\, dQ/d\theta\, |n - m, in\rangle\} \tag{15}$$

Let C_1 be any simple closed curve encircling the tube of quantal dynamical trajectories in extended phase space. $\theta = 0$ and $\theta = l$ are same points on C_1. Through any point on C_1, there are those dynamical trajectories of the motion. Choose another closed curve C_2 on this tube of trajectories such that it encircles this tube and intersects the generatrix of the tube once. Take the integral of the expression (15) with respect to θ along curves C_1 and C_2 (Gantmacher, 1970), one has

$$\oint_{c_1} \langle out, m|T^* \int_V d^3x(\pi_\alpha \Delta\psi^\alpha - \mathcal{H}_c \Delta t)|n - m, in\rangle$$
$$- \oint_{c_2} \langle out, m|T^* \int_V d^3x(\pi_\alpha \Delta\psi^\alpha - \mathcal{H}_c \Delta t)|n - m, in\rangle$$
$$+ \oint_c \langle out, m|T^* \int d^4x \left(\frac{\delta I^p}{\delta\psi^\alpha}\delta\psi^\alpha + \frac{\delta I^p}{\delta\pi_\alpha \delta n_\alpha}\right) |n - m, in\rangle$$
$$= i \oint_{c_k} \{[dF/d\theta + dG/d\theta] + \langle out, m|dQ/d\theta|n - m, in\rangle\} \tag{16}$$

Since $\theta = 0$ and $\theta = l$ are same points on the closed curve, along those closed curves, the integral of $f(\theta)$, $g(\theta)$, and $J_1(\theta)$ must be equal to zero. Due to m and n are arbitrary, we have

$$\oint_{c_1} T^* \left[\int_V d^3x(\pi_\alpha \Delta\psi^\alpha - \mathcal{H}_c \Delta t)\right] - \oint_{c_2} T^* \left[\int_V d^3x(\pi_\alpha \Delta\psi^\alpha - \mathcal{H}_c \Delta t)\right]$$
$$+ \oint_c T^* \left[\int d^4x \left(\frac{\delta I^p}{\delta\psi^\alpha}\delta\psi^\alpha + \frac{\delta I^p}{\delta\pi_\alpha}\delta\pi_\alpha\right)\right] = 0 \tag{17}$$

Now, we deduce the quantum canonical equations for this system, since

(Henneaux and Teitelboim, 1992)

$$\langle \psi'^{\alpha}, t'| \frac{\delta I^p}{\delta \psi^{\alpha}} |\psi^{\alpha}, t\rangle = \int \mathcal{D}\psi^{\alpha}\mathcal{D}\pi_{\alpha} \frac{\delta I^p}{\delta \psi^{\alpha}} \exp\{iI^P\} \tag{18a}$$

$$\langle \psi'^{\alpha}, t'| \frac{\delta I^p}{\delta \pi_{\alpha}} |\psi^{\alpha}, t\rangle = \int \mathcal{D}\psi^{\alpha}\mathcal{D}\pi_{\alpha} \frac{\delta I^p}{\delta \pi_{\alpha}} \exp\{iI^P\} \tag{18b}$$

for the arbitrary state $\left|\psi'^{\alpha}, t'\right\rangle$ and $|\psi^{\alpha}, t\rangle$, from the classical canonical equations ($\frac{\delta I^p}{\delta \psi^{\alpha}} = 0$, $\frac{\delta I^p}{\delta \pi_{\alpha}} = 0$), we can obtain

$$\langle \psi'^{\alpha}, t'| \frac{\delta I^p}{\delta \psi^{\alpha}} |\psi^{\alpha}, t\rangle = \langle \psi'^{\alpha}, t'| \frac{\delta I^p}{\delta \pi_{\alpha}} |\psi^{\alpha}, t\rangle = 0, \tag{19}$$

Because $\left|\psi'^{\alpha}, t'\right\rangle$ and $|\psi^{\alpha}, t\rangle$ are arbitrary, thus the quantum dynamical trajeories are determined by following quantal canonical equations

$$\frac{\delta I^p}{\delta \psi^{\alpha}} = 0, \qquad \frac{\delta I^p}{\delta \pi_{\alpha}} = 0 \tag{20}$$

Using the quantal canonical equations (20), one has

$$W = T^* \oint_c \int_V d^3x(\pi_{\alpha}\Delta\psi^{\alpha} - \mathcal{H}_c\Delta t) = inv \tag{21}$$

Therefore, with an arbitrary displacement and the deformation of the closed curve C along any tube of those dynamical trajectories, the integral W along the closed curve C is an invariance. W is called the Poincare-Cartan integral which is invariant for the system with a regular Lagrangian in field theories at the quantum level.

For a system with a singular Lagrangian, let $\Lambda_k(t, \psi^{\alpha}, \pi_{\alpha}) \approx 0(k = 1, 2, \ldots, a)$ be first-class constraints, and $\theta_i(t, \psi^{\alpha}, \pi_{\alpha}) \approx 0(i = 1, 2, \ldots, b)$ be second-class constraints. The gauge conditions connecting with the first-class constraints are $\Omega_l(t, \psi^{\alpha}, \pi_{\alpha}) \approx 0(l = 1, 2, \ldots, a)$. According Faddeeve-Senjanovic quantization formulation, the phase-space generating functional of Green function for the system with a singular Lagrangian is given by (Li and Jiang, 2002)

$$Z[J, K] = \int \mathcal{D}\psi^{\alpha}\mathcal{D}\pi_{\alpha} \prod_{i,k,l} \delta(\theta_i)\delta(\Lambda_k)\delta(\Omega_l) \det|\{\Lambda_k, \Omega_l\}| \, [\det|\{\theta_i, \theta_j\}|]^{1/2}$$
$$\cdot \exp\left\{i\left[I^P + \int_G d^4x(J_{\alpha}\psi^{\alpha} + K^{\alpha}\pi_{\alpha})\right]\right\} \tag{22}$$

Using the properties of the δ-function and the properties of the Grassmann variables $C_a(x)$ and $\bar{C}_a(x)$, the expression (22) can be written as

$$Z[J^\alpha, K_\alpha, \eta^m, \bar{j}, \bar{k}, j, k] = \int \mathcal{D}\varphi_\alpha \mathcal{D}\pi^\alpha \mathcal{D}\lambda_m \mathcal{D}\bar{C}_a \mathcal{D}\pi^a \mathcal{D}C_a \mathcal{D}\bar{\pi}^a \times \exp\left\{i \int d^3x \big(L^P_{eff} + J^\alpha \varphi_\alpha + K_\alpha \pi^\alpha + \eta^m \lambda_m + \bar{j}^a C_a + \bar{C}_a j^a + \bar{k}_a \pi^a + \bar{\pi}^a k_a\big)\right\} \tag{23}$$

where

$$\mathcal{L}^P_{eff} = \mathcal{L}^P + \mathcal{L}_m + \mathcal{L}_{gh} \tag{24}$$

$$\mathcal{L}^p = \pi_\alpha \dot{\psi}^\alpha - \mathcal{H}_c \tag{25}$$

$$\mathcal{L}_m = \lambda_i \theta_i + \lambda_k \Lambda_k + \lambda_l \Omega_l \tag{26}$$

$$L_g h = \int d^3y [\bar{C}_k(x)\{\Lambda_k(x), \Omega_l(y)\} C_l(y) + \frac{1}{2}\bar{C}_i(x)\{\theta_i(x), \theta_j(y)\} C_j(y)] \tag{27}$$

and $\lambda_m = (\lambda_k, \lambda_l, \lambda_i)$, λ_k, λ_i and λ_l are multiplier fields connected with the constraints Λ_k, θ_i and Ω_l, respectively. $\bar{\pi}^a(x)$ and $\pi^a(x)$ are canonical momenta conjugate to $C_a(x)$ and $\bar{C}_a(x)$, respectively, here we have introduced the exterior sources η^m, $\bar{j}^a$, $\bar{k}_a$, j^a, and k_a with respect to the fields $\lambda_m, C_a, \pi^a, \bar{C}_a$ and $\bar{\pi}^a$, respectively. Thus, the quantal canonical equations are determined by $\mathcal{H}_{eff} = \pi_\alpha \dot{\psi}^\alpha - \mathcal{L}^p_{eff}$. for the system with a singular Lagrangian. Hence, we can still proceed in the same way to obtain the PCII for the system with a singular Lagrangian when the Jaconbian of the transformation (3) is not equal to unity. But in the result for this case, one must use $\mathcal{H}_{eff}$ instead of $\mathcal{H}_c$ in expression (21)

$$W' = T^* \oint_c \int_V d^3x (\pi_\alpha \Delta\psi^\alpha - \mathcal{H}_{eff} \Delta t) = inv \tag{28}$$

The closed curves must satisfy all the constraint conditions. It can be shown that the quantal PCII for the system with a singular Lagrangian can also be derived when the effective action I^p_{eff} instead of I^p.

3. QUANTAL PC INTEGRAL INVARIANT AND QUANTAL CANONICAL EQUATIONS

Now, we first consider the discrete system. We can divide the space domain V into a very large number of small cells and use ΔV_i to denote the volume of i-th cell; ψ_i^α, the average of the variables $\psi^\alpha(x)$ on ΔV_i and $p_\alpha^i(t)$, the canonical momenta corresponding to ψ_i^α. Thus, $p_\alpha^i(t) = \pi_\alpha^i \Delta V_i$ (not summing up i). In this way the discrete case for expression (21) can be written as

$$W = T^* \oint_c \left(p_\alpha^i \Delta \psi_i^\alpha - H_c \Delta t \right) = inv \tag{29}$$

When $\Delta V_i \to 0$, $\psi_i^\alpha(t) \to \psi^\alpha(\vec{x}, t)$, $\pi_\alpha^i(t) \to \pi_\alpha(\vec{x}, t)$, the continuous limit of (29) converts into expression (21). Using this result, it is easy to extend the conclusion of the discrete system to the system in the field theories (Li, 1993).

In classical theories, it has been proved that PCII is equivalent to the classical motion equations (Benavent and Gomis, 1979; Dominici and Gomis, 1980; Li and Jiang, 2002). We can show that this equivalent relation at the quantum level still holds true. Now we study the inversion of the Section 2, i.e. the quantum equations can be derived from the quantal PCII for the system with regular/singular Lagrangian. Let it be first considered that the quantal motion equations of the discrete system in the phase space which is given by (Mei *et al.*, 1991) (similar to the analysis of (18)–(20), operators can be converted into classical numbers)

$$\dot{\psi}_i^\alpha = \frac{d\psi_i^\alpha}{dt} = Q_i^\alpha\left(t, \psi_i^\alpha, p_\alpha^i\right), \quad \dot{p}_\alpha^i = \frac{dp_\alpha^i}{dt} = P_\alpha^i\left(t, \psi_i^\alpha, p_\alpha^i\right) \tag{30}$$

From (21), we can obtain

$$\begin{aligned} 0 = \frac{d}{dt} W' &= \oint_c \left(\frac{dp_\alpha^i}{dt} \Delta \psi_i^\alpha + p_\alpha^i \frac{d}{dt} \Delta \psi_i^\alpha - \frac{dH_c}{dt} \Delta t - H_c \frac{d}{dt} \Delta t \right) \\ &= \oint_c \left[\frac{dp_\alpha^i}{dt} \left(\delta \psi_\alpha^i + \dot{\psi}_i^\alpha \Delta x^0 \right) + p_\alpha^i \frac{d}{dt} \left(\delta \psi_\alpha^i + \dot{\psi}_i^\alpha \Delta x^0 \right) - \frac{dH_c}{dt} \Delta t \right] \end{aligned} \tag{31}$$

Using integrating by parts, from (31), one obtains

$$\begin{aligned} 0 &= \oint_c \left[\frac{dp_\alpha^i}{dt} \delta \psi_i^\alpha + p_\alpha^i \delta \frac{d}{dt} \psi_i^\alpha - \frac{dH_c}{dt} \Delta t \right] \\ &= \oint_c \left[\frac{dp_\alpha^i}{dt} \delta \psi_i^\alpha - \frac{d\psi_i^\alpha}{dt} \delta p_\alpha^i - \frac{dH_c}{dt} \delta t \right] = 0 \end{aligned} \tag{32}$$

From Eq. (32), one obtains

$$\oint_c \left[P_\alpha^i \delta \psi_i^\alpha - Q_i^\alpha \delta p_\alpha^i - \frac{dH_c}{dt} \delta t \right] = 0 \tag{33}$$

Because the contour of the integrating is arbitrary, and then the integrand is the variation of $-H_c(t, \psi_i^\alpha, p_\alpha^i)$,

$$P_\alpha^i \delta \psi_i^\alpha - Q_i^\alpha \delta p_\alpha^i - \frac{dH_c}{dt} \delta t = -\delta H_c\left(t, \psi_i^\alpha, p_\alpha^i\right) \tag{34}$$

thus

$$P_{\alpha}^{i}=-\frac{\partial H_c}{\partial \psi_i^{\alpha}},\quad Q_i^{\alpha}=\frac{\partial H_c}{\partial p_{\alpha}^{i}},\tag{35}$$

It is to say that the equivalence between the quantal canonical equations and the quantal PC integral invariant is proved for the system with a regular Lagrangian.

One can still proceed in the same way to obtain the equivalence between the quantal canonical equations and the quantal PC integral invariant for the system with a singular Lagrangian. But in this case, $(\psi_i^{\alpha}, C_a^i, \bar{C}_a^i, \eta_i^m)$ and $(p_{\alpha}^i, p_i^a, \bar{p}_i^a)$ should be used instead of ψ_i^{α} and p_{α}^i, and H_{eff} should be used instead of H_c.

From the above discussion we can show that the necessary and sufficient condition for the equation of motion to be quantal canonical equations is that the PC integral be invariant at the quantum level.

When $\Delta V_i \to 0$, the continuous limit of (35) convert into (20), thus the equivalence between the quantal PCII and the quantal canonical equations of the discrete system can be extended to the system in the field theories.

4. THE QUANTAL PCII AND THE CANONICAL TRANSFORMATION

The canonical transformation in field theories can be stated as follows. Suppose the equations of motion of a dynamical system are given by Eq. (20). Then the canonical transformation is to be defined as such transformation of the canonical variables $\psi^{\alpha}, \pi_{\alpha}$.

$$\psi^{\alpha *}=Q'^{\alpha}(t,\psi^{\alpha},\pi_{\alpha}),\quad \pi_{\alpha}^{*}=P_{\alpha}'(t,\psi^{\alpha},\pi_{\alpha})\tag{36}$$

which leaves the form of Eq. (18) of the system to be invariant. If under transformation two function $H_c^*=\int_V d^3x H_c^*$ (for the system with a singular Lagrangian, H_{eff} should be used instead of H_c) and G exist so that

$$\int_V d^3x(\pi_{\alpha}\Delta\psi^{\alpha}-\mathcal{H}_c\Delta t)=\int_V d^3x(\pi_{\alpha}^{*}\Delta\psi^{\alpha *}-\mathcal{H}_c^{*}\Delta t)+\Delta G\tag{37}$$

Then transformation is canonical, and G is called generating function. In fact, one can choose a close curve in the extended phase space, from (37) one can obtain

$$\oint_c\left[\int_V d^3x(\pi_{\alpha}\Delta\psi^{\alpha}-\mathcal{H}_c\Delta t)-\int_V d^3x(\pi_{\alpha}^{*}\Delta\psi^{\alpha *}-\mathcal{H}_c^{*}\Delta t)\right]=0\tag{38}$$

If C^* is a closed curve obtained from C by means of the transformation (36), then (38) can be written as

$$\oint_c\left[\int_V d^3x(\pi_{\alpha}\Delta\psi^{\alpha}-\mathcal{H}_c\Delta t)=\oint_{c''}\int_V d^3x(\pi_{\alpha}^{*}\Delta\psi^{\alpha *}-\mathcal{H}_c^{*}\Delta t)\right]\tag{39}$$

Because the ψ^{α} and π_{α} satisfy the equations of motion (20), the left-hand side of Eq. (39) is PCII at the quantum level, i.e. the left-hand side of Eq. (39) is invariant with the displacement and deformation of the closed curve C along the tube of the dynamical trajectories given by the solution of Eq. (20). Therefore, the right-hand side of Eq. (39) will be invariant with the displacement of the closed curve C^* along the tube obtained by means of the transformation (36). That is to say , the right –hand side of Eq. (39) is also a PCII at the quantum level with respect to the transformed new variables. Thus, the transformed trajectories must satisfy the quantal canonical equations of the system, and therefore, the transformation (36) is canonical(Li, 1993).

5. DISCUSSION AND CONCLUSION

On the basis of the invariance of phase-space generating function of Green function, considering the transformation property in the extended phase space, along the quantal motion trajectories, the quantal PCII in field theory for a system with a regular/singular Lagrangian is derived. It is proved that the PCII is equivalent to the quantal canonical equations, thus the classical PCII is extended to the quantum level. For the singular system, the quantal PCII should be determined by the effective Hamiltonian H_{eff} not the canonical Hamiltionian $\mathcal{H}_c$, the H_{eff} involves all constraints and gauge conditions. This is different from the classical theories at all. In classical theories, the expressions of PCII for a regular system and a singular system are completely similar. The differences are that the variations of the canonical variables for a system with a regular Lagrangian are arbitrary but that of the system with a singular Lagrangian are restricted by the constraint conditions (the constraint conditions are invariant under the simultaneous variation). Both are canonical Hamiltonian $\mathcal{H}_c$ that involves in PCII at the classical level. This result also differs from the quantum case. The expressions of PCII for the system with a regular Lagrangian and the system with a singular Lagrangian are similar when $\Delta t = 0$ at quantum level.

The conserved quantities corresponding to the classical symmetries perhaps do not exist at the quantum level. The case is called quantum anomaly. For example, in the Noether theorem at the quantum level, due to the constraints for the system with a singular Lagrangian in the phase space, the effective Hamiltonian H_{eff} is different from the canonical one $\mathcal{H}_c$. Moreover, the quantal conserved quantities can be obtained if the effective canonical action is invariant under the global transformation in phase space and the Jacobian of the corresponding local transformation is equal to unity. In general, there is quantum anomaly when the Jacobian of the transformation is not equal to unity. But this case does not occur for the PCII. Even if the Jacobian of the transformation is not equal to unity, the PCII can also be derived. The case is different from the quantal first Noether theorem. The cause arises from the equivalence between the PCII and the quantal canonical equations.

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